

MATH 337 – Numerical Differential Equations

Part I: Initial-value problems for Ordinary Differential equations (ODEs).

1. Review. Euler method and its modifications for 1st-order ODEs.
 - Taylor series.
 - Existence and uniqueness theorem for ODEs.
 - Analytic solution of a linear, inhomogeneous, 1st-order ODE.
 - Simple (forward) Euler method.
 - Truncation error and global error of the Euler method.
 - Modifications of the Euler method: Modified Euler and Midpoint methods.
 - How to improve the global error: Romberg method.
2. Runge-Kutta (RK) methods for 1st-order ODEs.
 - Modified Euler and Midpoint methods as particular cases of RK methods.
 - Higher-order RK methods; the Classical RK method.
 - Estimation of the step size to achieve a prescribed accuracy; Runge-Kutta-Fehlberg method.
3. Multi-step methods; Predictor-corrector methods; Implicit methods for 1st-order ODEs.
 - The idea behind the multi-step methods.
 - Adams-Bashford, Leap-frog, and a 3rd-order divergent multi-step methods.
 - Predictor-corrector methods.
 - Implicit 1st and 2nd-order Euler methods.
4. Consistency, stability, and convergence of numerical methods for 1st-order ODEs.
 - Definitions and main theorem: “consistency+stability $\implies$ convergence”.
 - The concept of stability of the analytical solution of a 1st-order linear ODE.
 - Investigation of stability of methods introduced in previous lectures. Conditionally stable, absolutely unstable, and absolutely stable methods.
5. Higher-order ODEs and systems of ODEs.
 - Generalizations of the studied methods to systems of 1st-order ODEs.
 - Higher-order ODEs as systems of 1st-order ODEs.
 - Special discretization schemes for 2nd-order ODEs: Simple central-difference and higher-order (Numerov’s) methods.
 - Symplectic methods: simple examples (symplectic Euler and Verlet-Störmer methods), and why they (sometimes) work better than non-symplectic methods.
 - Stability of numerical methods for systems of ODEs and higher-order ODEs.
 - Stiff equations.

Part II: Boundary-value problems (BVP) for ODEs.

6. Introduction. (Few available theorems about) existence and uniqueness of solutions of BVPs.
7. The shooting method.
 - Shooting method for linear BVPs with Dirichlet and Neumann boundary conditions.
 - Multiple shooting method.
 - Shooting method for nonlinear BVPs.
8. Finite-difference methods for BVPs.
 - The matrix problem for the discretized solution of a *linear* BVP with the Dirichlet boundary conditions. Existence and uniqueness of solution to a matrix problem where the matrix is diagonally dominant. Gerschgorin circles and eigenvalues of a matrix.
 - Higher-order discretizations and an estimate of the error.
 - Other types of boundary conditions.
 - Iterative solutions of nonlinear BVPs.

9. Concepts behind Finite-Element methods.

- The collocation method.
- The Galerkin method.

10. Eigenvalue problems. (**not** covered)

- Definition of eigenvalues and eigenfunctions of BVPs.
- Issues in finding eigenvalues of large matrices.
- Householder transformation of a matrix to an upper-Hessenberg form.
- Power method and its modifications.
- The shooting method.

Part III: Partial-differential equations (PDEs).

11. Classification of PDEs.

- Classification of physical problems leading to PDEs.
- Classification of PDEs. The concept, and significance, of characteristics.

12. Parabolic PDEs in 1 spatial dimension (Heat equation): Simple explicit method and stability analyses.

- Formulation of the model problem and the general property of the analytical solution.
- The simple explicit method for the Heat equation.
- The matrix and von Neumann stability analyses.
- Explicit methods of higher order.
- Effect of smoothness of the initial condition on the order of consistency of a numerical method.

13. Implicit methods for the Heat equation.

- Derivation of the Crank-Nicolson (CN) scheme.
- Truncation error of the CN scheme.
- Stability analysis of the CN and related methods.
- Improving the accuracy of the CN method.

14. Numerical methods for generalizations of the one-dimensional Heat equation.

- Derivative (Neumann) boundary conditions.
- Equations with variable coefficients.
- Nonlinear parabolic equations.

15. Heat equation in 2 and 3 spatial dimensions.

- The explicit method and the CN method for the 2-D Heat equation.
- Criteria for a computationally-efficient scheme in 2 dimensions.
- Alternating-Direction Implicit (ADI) methods: Peaceman-Rachford, Douglas-Rachford, D'yakonov, and Craig-Sneyd methods.
- Boundary conditions for ADI methods.

16. Hyperbolic PDEs: Analytical solutions and characteristics.

17. Method of characteristics for solving hyperbolic PDEs.