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The eigenvalues of tridiagonal matrices Autumn 2009

Note TMA4205

Tridiagonal matrices are often found in connection with finite differences.

Tridiagonal matrices are easy to deal with since there exists efficient numerical methods both for solving their linear systems of equations and eigenvalue problem. Here we consider the eigenvalue problem for a general tridiagonal matrix of the form

$$A = \begin{bmatrix} a & b & & & \\ c & a & b & & \\ & c & \ddots & \ddots & \\ & & \ddots & a & b \\ & & & c & a \end{bmatrix} \in \mathbf{R}^{m \times m}.$$

We solve the eigenvalue problem

$$Ax = \lambda x,$$

where $\lambda \in \mathbf{R}$ and $x = [x_1, \dots, x_m]^T \neq 0$. We write out the eigenvalue problem for A to obtain the difference equation

$$\begin{aligned} cx_{j-1} + ax_j + bx_{j+1} &= \lambda x_j, & j = 1, \dots, m \\ x_0 = x_{m+1} &= 0 \end{aligned}$$

which is equivalent to

$$\begin{aligned} cx_{j-1} + (a - \lambda)x_j + bx_{j+1} &= 0, & j = 1, \dots, m \\ x_0 = x_{m+1} &= 0 \end{aligned}$$

You may remember from earlier exercises that the solution of such an equation can be expressed in terms of the roots of the characteristic polynomial, which in this case is

$$p(r) = br^2 + (a - \lambda)r + c.$$

So assume that the roots of p are given as r_1 and r_2 . Then the solution of the difference equation is

$$x_j = \alpha r_1^j + \beta r_2^j$$

for $j = 0, \dots, m + 1$. We determine the unknown coefficients by using the initial condition:

$$x_0 = \alpha + \beta = 0 \Leftrightarrow \beta = -\alpha,$$

which gives

$$x_j = \alpha(r_1^j - r_2^j), \quad j = 0, \dots, m + 1.$$

Furthermore we have

$$x_{m+1} = \alpha(r_1^{m+1} - r_2^{m+1}) = 0.$$

Since $x \neq 0$ we need $\alpha \neq 0$, so we find that

$$r_1^{m+1} = r_2^{m+1} \Leftrightarrow \left(\frac{r_1}{r_2}\right)^{m+1} = 1.$$

We can eliminate r_2 from this equation through the identity

$$\begin{aligned} r_1 r_2 &= \left(\frac{-(a-\lambda) + \sqrt{(a-\lambda)^2 - 4bc}}{2b}\right) \left(\frac{-(a-\lambda) - \sqrt{(a-\lambda)^2 - 4bc}}{2b}\right) \\ &= \frac{(a-\lambda)^2 - ((a-\lambda)^2 - 4bc)}{4b^2} \\ &= \frac{c}{b}. \end{aligned}$$

Thus

$$\left(\frac{r_1}{r_2}\right)^{m+1} = \left(\frac{r_1^2}{r_2 r_1}\right)^{m+1} = \left(\frac{r_1^2}{\frac{c}{b}}\right)^{m+1} = 1$$

The roots of a quadratic polynomial are in general complex, so the above equation can be written in the form

$$\frac{r_1^2}{\frac{c}{b}} = e^{2\pi i(\frac{s}{m+1})}, \quad s = 1, \dots, m.$$

We immediately see that the possible roots are

$$\begin{aligned} r_{1,s} &= \sqrt{\frac{c}{b}} e^{\pi i(\frac{s}{m+1})} \\ r_{2,s} &= \sqrt{\frac{c}{b}} e^{-\pi i(\frac{s}{m+1})}, \end{aligned}$$

where $s = 1, \dots, m$. For every $s = 1, \dots, m$ there is thus an eigenvalue λ_s given by the equation

$$\begin{aligned} r_{1,s} + r_{2,s} &= \frac{\lambda_s - a}{b} \\ &\Downarrow \\ \sqrt{\frac{c}{b}}(e^{\pi i(\frac{s}{m+1})} + e^{-\pi i(\frac{s}{m+1})}) &= \frac{\lambda_s - a}{b} \\ &\Downarrow \\ 2\sqrt{\frac{c}{b}} \cos\left(\frac{\pi s}{m+1}\right) &= \frac{\lambda_s - a}{b} \\ &\Downarrow \\ \lambda_s &= a + 2\sqrt{bc} \cos\left(\frac{\pi s}{m+1}\right) \end{aligned}$$

The corresponding eigenvector $x_{s,j}$ is then

$$\begin{aligned} x_{s,j} &= \alpha(r_{1,s}^j + r_{2,s}^j) \\ &= \alpha\left(\frac{c}{b}\right)^{j/2} (e^{\pi i(\frac{js}{m+1})} - e^{-\pi i(\frac{js}{m+1})}) \\ &= 2i\alpha\left(\frac{c}{b}\right)^{j/2} \sin\left(\frac{\pi js}{m+1}\right), \end{aligned}$$

i.e.

$$x_s = \left[\left(\frac{c}{b}\right)^{1/2} \sin\left(\frac{\pi s}{m+1}\right), \dots, \left(\frac{c}{b}\right)^{m/2} \sin\left(\frac{\pi m s}{m+1}\right) \right],$$

for $s = 1, \dots, m$.

EXAMPLE: Consider the eigenvalues of the matrix

$$A = I + rD,$$

where

$$D = \begin{bmatrix} -2 & 1 & & & \\ & 1 & \ddots & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & 1 & -2 \end{bmatrix} \in \mathbf{R}^{n \times n}.$$

Set $\lambda_s(A) = 1 + r\lambda_s(D)$ for $s = 1, \dots, n$, from the discussion above we then have

$$\lambda_s(D) = -2 + 2 \cos\left(\frac{\pi s}{n+1}\right) = -4 \sin^2\left(\frac{\pi s}{2(n+1)}\right).$$

Therefore,

$$\lambda_s(A) = 1 - 4r \sin^2\left(\frac{\pi s}{2(n+1)}\right)$$

for $s = 1, \dots, n$.