MATH 6737 – Numerical Differential Equations

Part I: Initial-value problems for Ordinary Differential equations (ODEs).

1. Review. Euler method and its modifications for 1st-order ODEs.
   - Taylor series.
   - Existence and uniqueness theorem for ODEs.
   - Analytic solution of a linear, inhomogeneous, 1st-order ODE.
   - Simple (forward) Euler method.
   - Truncation error and global error of the Euler method.
   - Modifications of the Euler method: Modified Euler and Midpoint methods.
   - How to improve the global error: Romberg method.
2. Runge-Kutta (RK) methods for 1st-order ODEs.
   - Modified Euler and Midpoint methods as particular cases of RK methods.
   - Higher-order RK methods; the Classical RK method.
   - Estimation of the step size to achieve a prescribed accuracy; Runge-Kutta-Fehlberg method.
3. Multi-step methods; Predictor-corrector methods; Implicit methods for 1st-order ODEs.
   - The idea behind the multi-step methods.
   - Adams-Bashford, Leap-frog, and a 3rd-order divergent multi-step methods.
   - Predictor-corrector methods.
   - Implicit 1st and 2nd-order Euler methods.
4. Consistency, stability, and convergence of numerical methods for 1st-order ODEs.
   - Definitions and main theorem: "consistency+stability $\Longrightarrow$ convergence".
   - The concept of stability of the analytical solution of a 1st-order linear ODE.
   - Investigation of stability of methods introduced in previous lectures. Conditionally stable, absolutely unstable, and absolutely stable methods.
5. Higher-order ODEs and systems of ODEs.
   - Generalizations of the studied methods to systems of 1st-order ODEs.
   - Higher-order ODEs as systems of 1st-order ODEs.
   - Special discretization schemes for 2nd-order ODEs: Simple central-difference and higher-order (Numerov's) methods.
   - Symplectic methods: simple examples (symplectic Euler and Verlet-Störmer methods), and why they (sometimes) work better than non-symplectic methods.
   - Stability of numerical methods for systems of ODEs and higher-order ODEs.
   - Stiff equations.

Part II: Boundary-value problems (BVP) for ODEs.

6. Introduction. (Few available theorems about) existence and uniqueness of solutions of BVPs.
7. The shooting method.
   - Shooting method for linear BVPs with Dirichlet and Neumann boundary conditions.
   - Multiple shooting method.
   - Shooting method for nonlinear BVPs.
8. Finite-difference methods for BVPs.
   - The matrix problem for the discretized solution of a *linear* BVP with the Dirichlet boundary conditions. Existence and uniqueness of solution to a matrix problem where the matrix is diagonally dominant. Gerschgorin circles and eigenvalues of a matrix.
   - Higher-order discretizations and an estimate of the error.
   - Other types of boundary conditions.
   - Iterative solutions of nonlinear BVPs.

9. Concepts behind Finite-Element methods.
    - The collocation method.
    - The Galerkin method.
10. Eigenvalue problems. (**not** covered)

Part III: Partial-differential equations (PDEs).

11. Classification of PDEs.
    - Clasification of physical problems leading to PDEs.
    - Clasification of PDEs. The concept, and significance, of charcteristics.
12. Parabolic PDEs in 1 spatial dimension (Heat equation): Simple explicit method and stability analyses.
    - Formulation of the model problem and the general property of the analytical solution.
    - The simple explicit method for the Heat equation.
    - The matrix and von Neumann stability analyses.
    - Explicit methods of higher order.
    - Effect of smoothness of the initial condition on the order of consistency of a numerical method.
13. Implicit methods for the Heat equation.
    - Derivation of the Crank-Nicolson (CN) scheme.
    - Truncation error of the CN scheme.
    - Stability analysis of the CN and related methods.
    - Improving the accuracy of the CN method.
14. Numerical methods for generalizations of the one-dimensional Heat equation.
    - Derivative (Neumann) boundary conditions.
    - Equations with variable coefficients.
    - Nonlinear parabolic equations.
15. Heat equation in 2 and 3 spatial dimensions.
    - The explicit method and the CN method for the 2-D Heat equation.
    - Criteria for a computationally-efficient scheme in 2 dimensions.
    - Alternating-Direction Implicit (ADI) methods: Peaceman-Rachford, Douglas-Rachford, D'yakonov, and Craig–Sneyd methods.
    - Boundary conditions for ADI methods.
16. Hyperbolic PDEs: Analytical solutions and characteristics.
17. Method of characteristics for solving hyperbolic PDEs.