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Similarly we may obtain the difference approximation for the boundary condition at $x = x_{N+1}$ as

$$u_N = u_{N+1} - hu'_{N+1} + h^2(\tfrac{1}{6}f_N + \tfrac{1}{3}f_{N+1}) \tag{6.161}$$

Substituting for u'_0 and u'_{N+1} from (6.158) into (6.160) and (6.161) the third boundary value problem may be replaced by the following equations

$$\left(1 + h\frac{a_1}{a_0}\right)u_0 - u_1 + h^2(\tfrac{1}{3}f_0 + \tfrac{1}{6}f_1) + \frac{h\gamma_1}{a_0} = 0$$
$$-u_{j-1} + 2u_j - u_{j+1} + h^2(\beta_0 u''_{j-1} + \beta_1 u''_j + \beta_2 u''_{j+1}) = 0, \quad 1 \leqslant j \leqslant N$$
$$-u_N + \left(1 + \frac{hb_1}{b_0}\right)u_{N+1} + h^2(\tfrac{1}{6}f_N + \tfrac{1}{3}f_{N+1}) - \frac{h\gamma_2}{b_0} = 0 \tag{6.162}$$

The system of nonlinear equations (6.162) is generally solved by the Newton-Raphson method discussed in Chapter 2.

6.6.3 Convergence of Difference Schemes

We shall now use some properties of the matrices given in Chapter 3 for establishing the convergence of the difference schemes for the numerical solution of the boundary value problem (6.137–6.138). The exact solution $u(x)$ of (6.156) satisfies

$$-u(x_{j-1}) + 2u(x_j) - u(x_{j+1}) + h^2(\beta_0 f(x_{j-1}, u(x_{j-1})) + \beta_1 f(x_j, u(x_j))$$
$$+\beta_2 f(x_{j+1}, u(x_{j+1}))) + T_j = 0, \quad 1 \leqslant j \leqslant n \tag{6.163}$$

where T_j is the truncation error.

Subtracting (6.163) from (6.156) and applying the Mean-Value Theorem, and substituting $\epsilon_j = u_j - u(x_j)$ we get the error equation

$$\mathbf{ME} = (\mathbf{J} + \mathbf{Q})\mathbf{E} = \mathbf{T} \tag{6.164}$$

matrix A from notes (8.44)

where

$$\mathbf{E} = [\epsilon_1\,\epsilon_2 \ldots \epsilon_N]^T, \quad \mathbf{T} = [T_1\,T_2 \ldots T_N]^T$$

$$\mathbf{Q} = h^2\begin{bmatrix} f_{u_1}\beta_1 & f_{u_2}\beta_1 & & \bigcirc \\ f_{u_1}\beta_0 & f_{u_2}\beta_1 & f_{u_3}\beta_2 & \\ & \ddots & \ddots & \ddots \\ \bigcirc & & f_{u_{N-1}}\beta_0 & f_{u_N}\beta_1 \end{bmatrix}$$

$$J = \begin{pmatrix} 2 & -1 & & 0 \\ -1 & 2 & -1 & \\ & \ddots & \ddots & \ddots \\ 0 & & & \end{pmatrix}$$

Thus we note from (6.164) that the convergence of the difference schemes depends on the properties of the matrix $\mathbf{M}$. We now show that the matrix $\mathbf{M} = \mathbf{J} + \mathbf{Q}$ is an irreducible, monotone matrix such that $\mathbf{M} \geqslant \mathbf{J}$ and $\mathbf{Q} \geqslant 0$. Since $\beta_\nu > 0$, $\nu = 0, 1, 2$ and $f_{u_j} > 0$, $j = 1(1)N$, we have $\mathbf{Q} > 0$ and hence

$$\mathbf{M} = \mathbf{J} + \mathbf{Q} > \mathbf{J}$$

It follows that

$$0 < \mathbf{M}^{-1} < \mathbf{J}^{-1} \tag{6.165}$$

From (6.164), we have

$$\|\mathbf{E}\| \leqslant \|\mathbf{M}^{-1}\|\,\|\mathbf{T}\| \leqslant \|\mathbf{J}^{-1}\|\,\|\mathbf{T}\| \tag{6.166}$$

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In order to simplify (6.166) further we determine $\mathbf{J}^{-1} = (j_{i,j})$ explicitly. On multiplying the rows of $\mathbf{J}$ by the jth column of $\mathbf{J}^{-1}$, we have the following equations:

(i) $2j_{1,j} - j_{2,j} = 0$

(ii) $-j_{i-1,j} + 2j_{i,j} - j_{i+1,j} = 0, \quad 2 \leqslant i \leqslant j-1$

(iii) $-j_{j-1,j} + 2j_{j,j} - j_{j+1,j} = 1$ (6.167)

(iv) $-j_{i-1,j} + 2j_{i,j} - j_{i+1,j} = 0, \quad j+1 \leqslant i \leqslant N-1$

(v) $-j_{N-1,j} + 2j_{N,j} = 0$

The solution of (6.167ii), using (6.167i), is given by

$$j_{i,j} = c_2 i, \quad 2 \leqslant i \leqslant j-1 \qquad (c_2 = j_{1,j}) \tag{6.168}$$

where c_2 is independent of i, but may depend on j. Similarly the solution of (6.167iv), using (6.167v), is given by

$$j_{i,j} = c_1\left(1 - \frac{i}{N+1}\right), \quad j+1 \leqslant i \leqslant N-1 \tag{6.169}$$

The constant c_1 depends only on j. On equating the expression for $j_{i,j}$ obtained from (6.168) and (6.169) for $i = j$, we get

$$c_2 j = c_1\left(1 - \frac{j}{N+1}\right) \tag{6.170}$$

Also, on substituting the values of $j_{i,j} (i = j-1, j+1)$ obtained from (6.168) and (6.169) in (6.167iii), we have

$$c_2 + \frac{c_1}{N+1} = 1 \tag{6.171}$$

Finally from (6.171) and (6.170), we get

$$c_1 = j, \quad c_2 = \frac{N-j+1}{N+1} \tag{6.172}$$

On substituting the values of c_1 and c_2, we have

$$j_{i,j} = \begin{cases} \dfrac{i(N-j+1)}{(N+1)}, & i \leqslant j \\ \dfrac{j(N-i+1)}{N+1}, & i \geqslant j \end{cases} \tag{6.173}$$

From (6.173) we see that $\mathbf{J}^{-1}$ is symmetric.

The row sum of $\mathbf{J}^{-1}$ is given as

$$\sum_{j=1}^{N} j_{i,j} = \frac{i(N-i+1)}{2} = \frac{(x_i - a)(b - x_i)}{2h^2}$$