11 Classification of partial differentiation equations (PDEs)

In this lecture, we will begin studying differential equations involving more than one independent variable. Since they involve partial derivatives with respect to these variables, they are called *partial* differential equations (PDEs). Although this course is concerned with numerical methods for solving such equations, we will first need to provide some analytical background on where those equations arise and how their setup is different from that of ODEs. This will be done in this lecture, while the subsequent lectures, except Lecture 16, will deal with the numerical methods proper.

11.1 Classification of physical problems described by PDEs

The majority of problems in physics and engineering fall into one of the following categories: (i) equilibrium problems, (ii) eigenvalue problems, and (iii) evolution problems.

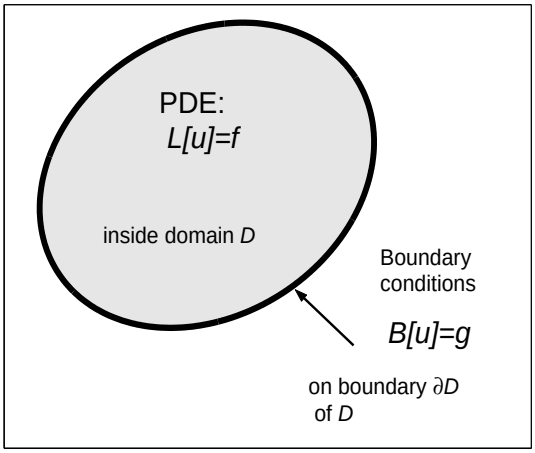
(i) Equilibrium problems are those where a steady-state spatial distribution of some quantity u inside a given domain D is to be determined by solving a differential equation

$$L[u] = f(x,y), \qquad (x,y) \in D$$

subject to the boundary condition

$$B[u] = g(x,y), \qquad (x,y) \in \partial D,$$

where ∂D is the boundary of D. Here L is a differential operator involving derivatives with respect to x and y (for the case of two spatial dimensions); B, in general, may also involve derivatives. These BVPs generalize, to two or more dimensions, the one-dimensional BVPs we studied in Lectures 6 through 9.



Examples of equilibrium problems include: Steady flows of liquids and gases; steady temperature distributions; equilibrium stress distributions in elastic structures.

(ii) Eigenvalue problems are extensions of equilibrium problems with *no external forces* where nontrivial (i.e. not identically zero) steady-state distributions exist only for special values of certain parameters, called eigenvalues. These eigenvalues, denoted λ, are to be determined along with the steady-state distributions themselves. The simplest form of an eigenvalue problem is

$$L[u] = \lambda u \quad \text{for} \quad (x,y) \in D; \qquad B[u] = 0 \quad \text{for} \quad (x,y) \in \partial D.$$

In a more complex setup, the eigenvalue may enter into the PDE, and even into the boundary condition, in a more complicated way.

Examples of eigenvalue problems include: Natural frequencies of vibrating strings and beams; resonances in electric circuits, mechanics, and acoustics; energy levels in quantum mechanics.

(iii) Evolution problems are extensions of initial value problems, where the distribution of a quantity u is not steady but exhibits transient behavior. Generally, the problem is to predict the evolution of the system at any time given its initial state. This is done by solving the PDE

$$L[u] = f(x,t), \quad x \in D \quad \text{for} \quad t > t_0,$$

given the initial state

$$I[u] = h(x), \quad x \in D \quad \text{for} \quad t = t_0,$$

and the boundary conditions

$$B[u] = g(x,t), \quad x \in \partial D \quad \text{and} \quad t \geq t_0.$$

The differential operator L now involves derivatives with respect to x and t.

Examples of evolution problems include: Propagation of waves of any nature; diffusion of a substance in a room; cooling down or heating an object.

For example, the mathematical problem of determining the evolution of a temperature distribution $u(x,t)$ inside a rod of length 1 is set up as follows:

$$L[u] = f(x,t), \quad 0 < x < 1, \quad t > 0,$$

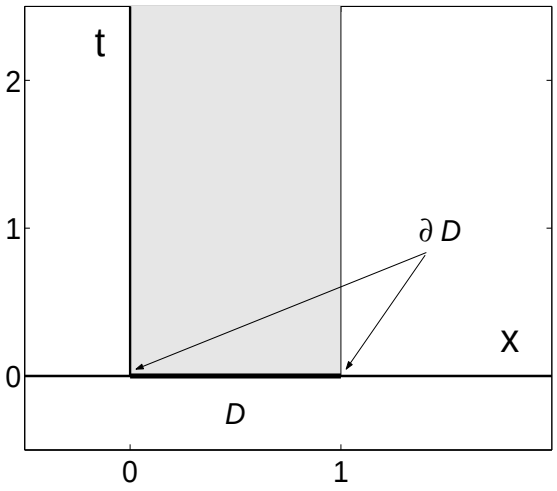
where the form of the operator L will be specified later;

$$u(x, t=0) = h(x), \quad 0 \leq x \leq 1,$$

where $h(x)$ is the initial temperature distribution inside the rod;

$$u(x=0,t) = g_0(t), \quad u(x=1,t) = g_1(t), \quad t \geq 0,$$

where $g_{0,1}(t)$ are the temperature values maintained at the two ends of the rod.



In subsequent lectures, we will consider exclusively evolution problems. To that end, we would like to obtain an unambiguous, mathematically rigorous criterion which allows one to distinguish problems of different categories. This is done in the next subsection.

11.2 Classification of PDEs into three types; characteristics

Here we will consider the question of how many initial or boundary conditions can or should be specified for a PDE, and where (in the (x,y)-space or (x,t)-space) it can or should be specified. We will concentrate on the case of two-dimensional spaces; generalizations to three- and four- (i.e., the time plus three spatial dimensions) dimensional cases are possible and for the most part straightforward. For definiteness, let us speak about the (x,y)-space until otherwise is indicated. (That is, for now, y may denote either the second spatial variable or the time variable t.)

As a reference, let us recall the situation with ODEs and, for concreteness, consider a second-order ODE $u'' = f(x,u,u')$. There, we could either specify the initial values for the dependent function u and its derivative u' at one point $x = x_0$, or the values of u (or more complicated expressions, e.g., (8.29)) at two points, $x = a$ and $x = b$. In the former case, we

have an IVP, and in the latter case, a BVP. Since, as we said earlier, we will be concerned with the evolution problems, which are higher-dimensional counterparts of IVPs, we proceed to recall how we were able to solve (conceptually, not technically) an IVP

$$u'' = f(x, u, u'), \qquad u(x_0) = u_0, \quad u'(x_0) = v_0\,. \tag{11.1}$$

Question: What does it actually mean "to solve an IVP"?
Answer: Obviously, it means to find the solution at every point x. But how? Well, we do so by "marching" from one point to the next: from x_0 to $x_0 + h$, then from $x_0 + h$ to $x_0 + 2h$, etc. (Here h is not related to a numerical step size (since we are discussing the analytical solution of the problem) but merely denotes a small increment.)

Now, to march from x_0 to $x_0 + h$, we use the Taylor expansion:

$$u(x_0 + h) = u(x_0) + hu'(x_0) + \frac{1}{2}h^2u''(x_0) + \frac{1}{6}h^3u'''(x_0) + \dots\,. \tag{11.2}$$

The first two terms on the r.h.s of (11.2) are known from the initial condition; the third term is known from the ODE, which we assume to be satisfied at $x = x_0$. The last term in (11.2) can then be found from

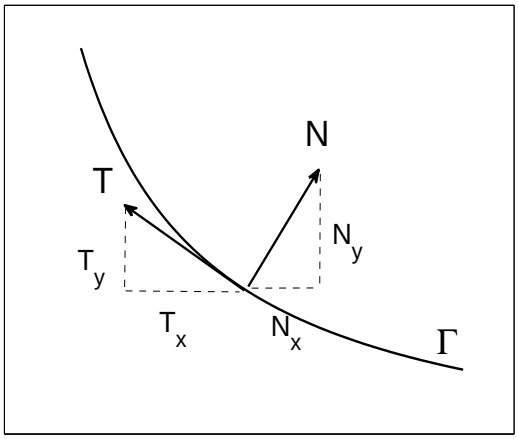
$$u'''(x) = \frac{du''}{dx} = \frac{df}{dx} = \frac{\partial f}{\partial x} + u'\frac{\partial f}{\partial u} + u''\frac{\partial f}{\partial u'}. \tag{11.3}$$

All omitted higher-order terms in (11.2) can be found analogously to (11.3). In this way, from (11.2) one finds $u(x_0 + h)$ at a point $x_0 + h$ that is sufficiently close to x_0. Then, to march on from $x_0 + h$ to $x_0 + 2h$, we need to know also $u'(x_0 + h)$, $u''(x_0 + h)$, $u'''(x_0 + h)$, etc., so as to be able to write a counterpart of (11.2) for $u(x_0 + 2h)$. The first derivative is found from the differentiated form of (11.2):

$$u'(x_0 + h) = u'(x_0) + hu''(x_0) + \frac{1}{2}h^2u'''(x_0) + \frac{1}{6}h^3u''''(x_0) + \dots \tag{11.4}$$

where, as we have explained above, we know every term on the r.h.s. The second derivative $u''(x_0 + h)$ is then found from the differential equation in (11.1) evaluated at $x = x_0 + h$. Next, $u'''(x_0 + h)$ is found from (11.3) evaluated at $x = x_0 + h$, and so on. We can thus march on in this way, one small (and otherwise arbitrary) step at a time, and thus find the solution for all x.

When we move from one independent variable (as in ODEs) to two (as in PDEs), it is intuitive to suppose that now the initial and/or boundary conditions should be specified along certain curves in the (x, y)-space rather than at a point. (In that case, the dimensions of both the differential equation and the initial/boundary condition are each increased by one.) Thus, let us assume that we know the dependent function u along some curve Γ in the (x, y)-plane and also the derivative $\partial u/\partial \vec{N}$ in the direction normal to Γ:



$$\begin{cases} u(x, y) = \tilde{g}_0(x, y), \\ \dfrac{\partial u(x, y)}{\partial \vec{N}} = \tilde{G}_1(x, y), \end{cases} \qquad (x, y) \in \Gamma, \tag{11.5}$$

where $\tilde{g}_0$ and $\tilde{G}_1$ are some known functions.

These equations can be rewritten in a different form that emphasizes that x and y in (11.5) are *not* independent. Indeed, recall that any curve Γ can be defined by parametric equations:

$$\Gamma: \qquad x = x(s), \quad y = y(s). \tag{11.6}$$

It is convenient to chose the parameter s to be the arclength along Γ (starting from an arbitrary point (a, b) on that curve). Then conditions (11.5) take on the form

$$\begin{cases} u = g_0(s), \\ u_x N_x + u_y N_y = G_1(s), \end{cases} \qquad (x(s), y(s)) \in \Gamma. \tag{11.5$'$}$$

where $g_0(s) \equiv \tilde{g}_0(x(s), y(s))$, $G_1(s) \equiv \tilde{G}_1(x(s), y(s))$. In the last equation in (11.5$'$), subscripts denote partial differentiation; i.e., $u_x \equiv \partial u/\partial x$ and $u_y \equiv \partial u/\partial y$, N_x, N_y are the components of $\vec{N}$ (see the figure above), and we have used the definition of the directional derivative (along vector $\vec{N}$) from Calculus III.

For reasons which will become clear later, we will now rewrite (11.5$'$) in yet another equivalent form which will explicitly exhibit the first partial derivatives of u. For that, take the directional derivative of the first equation in (11.5$'$) along the tangent vector $\vec{T}$:

$$u_x T_x + u_y T_y = dg_0(s)/ds, \tag{11.7}$$

where $T_{x,y}$ are the components of $\vec{T}$. Then from (11.7) and the last equation in (11.5$'$) one can solve for u_x and u_y separately at each point on Γ. Thus, Eqs. (11.5) are equivalent to

$$\begin{cases} u = g_0(s), \\ u_x = g_1(s), \\ u_y = g_2(s), \end{cases} \qquad (x(s), y(s)) \in \Gamma, \tag{11.8}$$

where $g_{1,2}$ are some linear combinations of dg_0/ds and G_1.

In the remainder of this course, we will consider PDEs of the form

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + F = 0, \tag{11.9}$$

where coefficients A, B, C may depend on any or all of x, y, u, u_x, u_y and coefficients D, E, F may depend on x, y, u. Given the PDE (11.9) and the initial/boundary conditions (11.8) on curve Γ, we would like to determine u at some point that is sufficiently near Γ. So, let (x_0, y_0) be some point on Γ and $(x_0+h, y_0+\kappa)$ with $h, \kappa \ll 1$ be a nearby point where we want to determine u. The **fundamental question** that we now ask is: *What are the restrictions on curve Γ and on the coefficients A, B, C in (11.9), under which one <u>can</u> determine $u(x_0 + h, y_0 + \kappa)$?*

We begin answering this question by writing the Taylor expansion for $u(x_0 + h, y_0 + \kappa)$ near point (x_0, y_0) on Γ, where we know both u and its first derivatives from (11.8):

$$\begin{aligned} u(x_0 + h, y_0 + \kappa) \;=\; & u(x_0, y_0) + h u_x(x_0, y_0) + \kappa u_y(x_0, y_0) + \\ & \frac{1}{2} h^2 u_{xx}(x_0, y_0) + h\kappa u_{xy}(x_0, y_0) + \frac{1}{2}\kappa^2 u_{yy}(x_0, y_0) + \\ & \frac{1}{6} h^3 u_{xxx}(x_0, y_0) + \ldots \end{aligned} \tag{11.10}$$

This expansion is the analog of (11.2). Now, as we have said, all terms on the r.h.s. of the first line of (11.10) are known from (11.8). If each of the three terms in the second line of (11.10)

can be found *separately* (i.e., as opposed to in the combination in which they enter Eq. (11.9)), then all the higher-order terms in expansion (11.10) can be found similarly to (11.3). Indeed, suppose we have found expressions for u_{xx}, u_{xy}, and u_{yy} in the form that generalizes (11.1) to two variables:

$$u_{xx} = f_1(x,y,u,u_x,u_y), \qquad u_{xy} = f_2(x,y,u,u_x,u_y), \qquad u_{yy} = f_3(x,y,u,u_x,u_y). \tag{11.11}$$

Then the third-order partial derivatives (see the last line in (11.10)) can be computed using the Chain Rule for a function of several variables. For example,

$$\begin{aligned} u_{yyy} &= \left.\frac{\partial u_{yy}}{\partial y}\right|_{x=\text{const}} = \left.\frac{df_3(x,y,u(x,y),u_x(x,y),u_y(x,y))}{dy}\right|_{x=\text{const}} \\ &= \frac{\partial f_3}{\partial y} + \frac{\partial f_3}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f_3}{\partial u_x}\frac{\partial u_x}{\partial y} + \frac{\partial f_3}{\partial u_y}\frac{\partial u_y}{\partial y} \;\equiv\; \frac{\partial f_3}{\partial y} + \frac{\partial f_3}{\partial u}u_y + \frac{\partial f_3}{\partial u_x}u_{xy} + \frac{\partial f_3}{\partial u_y}u_{yy}\,. \end{aligned} \tag{11.12}$$

The first two terms on the r.h.s. of (11.12) are known from (11.8). Therefore, if we also know u_{xy} and u_{yy} on Γ, we then can compute the last two terms in (11.12) and hence the u_{yyy}. Other third- and higher-order derivatives in the Taylor expansion (11.10) can be computed analogously. Thus, will be able to find $u(x_0+h, y_0+\kappa)$ if and only if we know u_{xx}, u_{xy}, and u_{yy} on Γ.

Now, we need three equations to be able to uniquely determine the three quantities u_{xx}, u_{xy}, and u_{yy}. The first equation is the PDE (11.9). The other two equations are found by taking the directional derivative of the last two equations of (11.8) along the tangent vector $\vec{T}$:

$$u_{xx}T_x + u_{xy}T_y = dg_1/ds\,, \tag{11.13}$$

$$u_{yx}T_x + u_{yy}T_y = dg_2/ds\,. \tag{11.14}$$

Recall that the subscripts of u denote corresponding partial derivatives, while T_x and T_y denote the x- and y-components of $\vec{T}$, as in (11.7).

Further, *if we assume u to be continuously differentiable at least twice*, then

$$u_{xy} = u_{yx}, \tag{11.15}$$

and Eqs. (11.9) and (11.13), (11.14) can be written as a linear system for u_{xx}, u_{xy}, and u_{yy} at any point on Γ:

$$\begin{pmatrix} A & 2B & C \\ T_x & T_y & 0 \\ 0 & T_x & T_y \end{pmatrix} \begin{pmatrix} u_{xx} \\ u_{xy} \\ u_{yy} \end{pmatrix} = \begin{pmatrix} -Du_x - Eu_y - F \\ dg_1/ds \\ dg_2/ds \end{pmatrix}. \tag{11.16}$$

This system yields a unique solution for u_{xx}, u_{xy}, and u_{yy} provided that the coefficient matrix is nonsingular. The matrix would be singular if its determinant vanishes:

$$\begin{aligned} A\,T_y^2 \,-\, 2B\,T_x\,T_y \,+\, C\,T_x^2 \,=\, 0 \qquad &\Rightarrow \\ A\left(\frac{dy}{dx}\right)_\Gamma^2 - 2B\left(\frac{dy}{dx}\right)_\Gamma + C = 0\,, \end{aligned} \tag{11.17}$$

where

$$\left(\frac{dy}{dx}\right)_\Gamma = \frac{T_y}{T_x} \tag{11.18}$$

is the slope of curve Γ (at a given point).

Thus, we have obtained **the answer to the fundamental question** posed above. Namely, if the initial/boundary conditions are prescribed on a curve Γ whose tangent at any point satisfies Eq. (11.17), then the corresponding initial-boundary value problem (IBVP) (11.5) and (11.9) *cannot* be solved for a twice-countinuously differentiable (see (11.15)) function $u(x,y)$. If the initial/boundary conditions (11.5) are prescribed along any other curve, the IBVP *can* be solved. Alternatively, the IBVP can still be solved if a smaller set of initial/boundary conditions (say, just the first line in (11.5)) is specified along Γ, or if u_{xy} (or any lower-order derivative of u) is allowed to be discontinuous across Γ.

Equation (11.17) gives one the mathematically rigorous criterion that separates all PDEs (11.9) into three types depending on the relation among A, B, and C.

$\underline{B^2 - AC < 0}$

In this case, no real solution for the slope $(dy/dx)_\Gamma$ can be found from the quadratic equation (11.17). This means that one can specify the initial/boundary conditions (11.5) along *any* curve in the plane, and be able to obtain the solution u *sufficiently close* to that curve. Such equations are called **elliptic**. Physical problems leading to elliptic equations are the equilibrium and eigenvalue problems, described in Sec. 11.1. Typical examples of such problems are the Laplace and Helmholtz equations:

$$u_{xx} + u_{yy} = 0, \qquad \text{(Laplace)}$$

$$u_{xx} + u_{yy} = \lambda u. \qquad \text{(Helmholtz)}$$

The boundary conditions for elliptic equations are usually imposed along the boundary of a closed domain D, as in the first figure in Sec. 11.1. One can also show that to obtain the solution inside the *entire* domain D rather than only "sufficiently close" to its boundary ∂D, one needs to impose only one of the conditions (11.5), but not both. On this remark, we leave the elliptic equations and will not consider them again in this course.

$\underline{B^2 - AC > 0}$

In this case, two real solutions for the slopes of curve Γ exist:

$$\left(\frac{dy}{dx}\right)_\Gamma = \frac{B \pm \sqrt{B^2 - AC}}{A}\,. \qquad (11.19)$$

These slopes specify two *distinct* directions in the (x,y)-plane, called **characteristics**. The correspondinf PDEs are called **hyperbolic**. Physical problems that lead to hyperbolic equations are the evolution problems dealing with propagation of waves (e.g., light or sound). The coordinates in this case are x, the spatial coordinate of propagation, and t, the time, rather than the second spatial coordinate y. The typical example is the Wave equation:

$$u_{xx} - u_{tt} = 0\,. \qquad \text{(Wave)}$$

The importance of characteristics in hyperbolic problems is two-fold: (i) the initial data for a smooth solution *cannot* be prescribed on a characteristic, and (ii) initial disturbances propagate along the characteristics. We will consider this latter issue in more detail when we begin to study numerical methods for hyperbolic PDEs.

$\underline{B^2 - AC = 0}$

In this case, only one value of the slope of Γ exists:

$$\left(\frac{dy}{dx}\right)_\Gamma = \frac{B}{A}\,. \qquad (11.20)$$

This gives only one direction of characteristics. The corresponding PDEs are called **parabolic**. Physical problems that lead to parabolic equations are usually diffusion-type problems. The typical example is the Heat equation,

$$u_{xx} - u_t = 0\,, \qquad \text{or} \qquad u_t = u_{xx}\,, \tag{Heat}$$

which describes, e.g., evolution of temperature inside a rod.

Since in the next four lectures we will consider methods of numerical solution of the Heat equation, let us discuss how boundary conditions can or should be set up for it. In fact, this was considered in the example at the end of Sec. 11.1. Namely, the initial condition for the Heat equation on $x \in [0,\, 1]$ is

$$u(x, t=0) = u_0(x), \quad 0 \le x \le 1, \qquad \left(\begin{array}{c}\text{Initial condition}\\ \text{for Heat equation}\end{array}\right)$$

and the boundary conditions are

$$u(0,t) = g_0(t), \quad u(1,t) = g_1(t), \qquad t \ge 0\,. \qquad \left(\begin{array}{c}\text{Boundary conditions}\\ \text{for Heat equation}\end{array}\right)$$

Note that the initial condition is prescribed along a characteristic! Indeed, for the Heat equation, $A = 1$, $B = C = 0$, and Eq. (11.20) gives the slope of characteristic as $dt/dx = 0$, which means that any line $t = \text{const}$ is a characteristic. The above, however, does *not* contradict the results of analysis of this subsection, because the initial condition corresponds only to the first equation in (11.5), while the second equation is absent. Thus, one *cannot* prescribe the rate of change u_t at the initial moment for the Heat equation.

11.3 Questions for self-assessment

1. Give examples from physics of equilibrium, eigenvalue, and evolution problems.
2. Explain how system (11.16) is set up (i.e., where its equations come from).
3. What is the significance of characteristics?
4. What types of physical problems lead to elliptic, hyperbolic, and parabolic equations?
5. How many characteristics does the Wave equation have?
6. Why does prescribing initial data on a characteristic for the Heat equation not prevent one from finding the solution of that IBVP?