16 Hyperbolic PDEs: Analytical solutions and characteristics

Hyperbolic PDEs describe propagation of disturbances in space and time when the total energy of the disturbances remains conserved. It is the condition of energy conservation that makes the hyperbolic equations different from parabolic ones, considered in Lectures 12 through 15. The following analogy with ODEs is intended to clarify the difference between hyperbolic and parabolic PDEs. Parabolic equations are multi-dimensional counterparts of the ODE

$$y' = -\lambda y, \qquad \mathrm{Re}\,\lambda > 0, \tag{16.1}$$

and thus describe processes of *relaxation* of the initial disturbance towards an equilibrium (which is $y = 0$ in the case of (16.1)). Hyperbolic equations are multi-dimensional counterparts of the ODE

$$y'' = -\lambda^2 y, \qquad \lambda^2 > 0, \tag{16.2}$$

which describes oscillations (see Lecture 5). However, hyperbolic PDEs describe not only oscillations, but also (and, in fact, much more often) propagation of initial disturbances. Examples include, e.g., propagation of sound and light.

16.1 Solution of the Wave equation

In fact, the basic form (i.e., before any perturbations or specific details are included into the model) of the equation that governs propagation of light and sound is the same. That same equation, called the Wave equation, also arises in a great variety of applications in physics and engineering. A classic example, considered in most textbooks, is the vibration of a string. The corresponding equation is

$$u_{tt} \;=\; c^2 u_{xx}\,. \tag{16.3}$$

In the above example of a string, $c = \sqrt{T/\rho}$, where T and ρ are the string's tension and density, respectively. As we will see shortly, in general, c is the speed of propagation of initial disturbances (e.g., the speed of sound for sound waves or the light speed for light waves).

To solve Eq. (16.3), we need to supplement it with initial and boundary conditions. We will do so later on. For now, let us discuss the general solution of (16.3). We will use this analytic solution as a reference for numerical solutions that we will obtain in Lecture 17.

Rewriting (16.3) in the form presented in Lecture 11:

$$1 \cdot u_{tt} \,+\, 2 \cdot 0 \cdot u_{xt} \,+\, (-c^2) \cdot u_{xx} \;=\; 0,$$

and then using Eq. (11.15), we obtain the equations for the two characteristics of (16.3):

$$\left(\frac{dx}{dt}\right)_{1,2} = \pm c\,. \tag{16.4}$$

Thus,

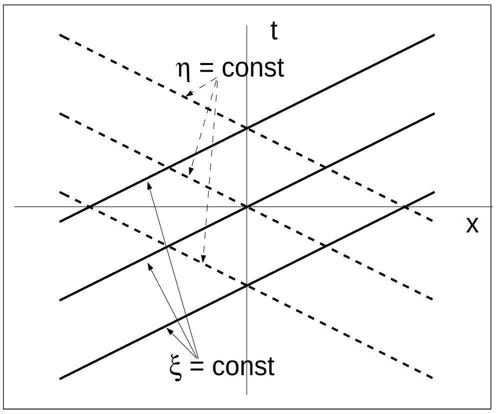
Along characteristics 1:

$$x - ct \equiv \xi \;=\; \text{const}; \tag{16.5a}$$

Along characteristics 2:

$$x + ct \equiv \eta \;=\; \text{const}. \tag{16.5b}$$

This is illustrated in the figure on the right.



The significance of the characteristics follows from the fact that any piece of initial or boundary data propagates along the characteristics and thereby determines the solution of (16.3) at any point in space and time. We will derive parts of this result later, and you will be asked to complete that derivation in a homework problem. For now, it will be sufficient for our purposes to give the general solution of (16.3) without a derivation:

$$u(x,t) = F(x - ct) + G(x + ct). \tag{16.6}$$

Here F and G are functions determined by the initial and boundary conditions, as we will show shortly. The meaning of solution (16.6) is this: The solution of the Wave equation splits into two waveforms, each of which travels along its own characteristic.

Now let us show how F and G are found assuming that the intial conditions

$$u(x, t = 0) = \phi(x), \qquad u_t(x, t = 0) = \psi(x), \qquad -\infty < x < \infty \tag{16.7}$$

are prescribed on the infinite line. That is, for now we will assume no explicit boundary conditions; implicitly, we will assume that there is no disturbance coming into the region of finite x-values from either $x = -\infty$ or $x = +\infty$. Note that in (16.7), $\phi(x)$ can be interpreted as the initial shape of the disturbance and $\psi(x)$, as its initial velocity. By substituting (16.6) into (16.7) and following a calculation outlined in Appendix 1, one obtains:

$$u(x,t) \;=\; \frac{1}{2}\left(\, \phi(x - ct) + \phi(x + ct)\,\right) \;+\; \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\, ds\,. \tag{16.8}$$

This formula is called the D'Alambert solution of the Wave equation (16.3) set up on the infinite line with the initial conditions (16.7). For example, when the initial velocity is zero everywhere, the solution (16.8) at any time is given by two replicas of the initial disturbance $\phi(x)$ that travel along the characteristics $x - ct = \text{const}$ and $x + ct = \text{const}$. Note especially that if the initial disturbance is not smooth (e.g., is discontinuous), the discontinuities are *not* smoothened out during the propagation but simply propagate along the characteristics.

Now let us show how formula (16.6) can be used to obtain a solution of (16.3) on a finite interval. Instead of initial conditions (16.7), we will now consider initial conditions

$$u(x, t = 0) = \phi(x), \qquad u_t(x, t = 0) = \psi(x), \qquad 0 \le x \le L \tag{16.9}$$

along with boundary conditions

$$u(x = 0, t) = g_0, \qquad u(x = L, t) = g_L, \qquad t > 0\,. \tag{16.10}$$

Note that the boundary values for u_t need not to be specified because they are determined by (16.10). Also, on physical grounds, we require that the boundary and initial conditions match:

$$\begin{array}{ll} \phi(x=0)=g_0(t=0), & \psi(x=0)=g_0'(t=0), \\ \phi(x=L)=g_L(t=0); & \psi(x=L)=g_L'(t=0). \end{array} \tag{16.11}$$

In what follows we illustrate the method of finding a solution of (16.3) on a finite interval for the special case when the boundary values g_0 and g_L do not depend on time. (The same method, but with additional effort, can be used in the general case of time-dependent boundary conditions.) When the boundary conditions are time-independent, we will first show that they can be set to zero without loss of generality. Using a trick analogous to that used in the homework problems for Lecture 9, we consider a modified function

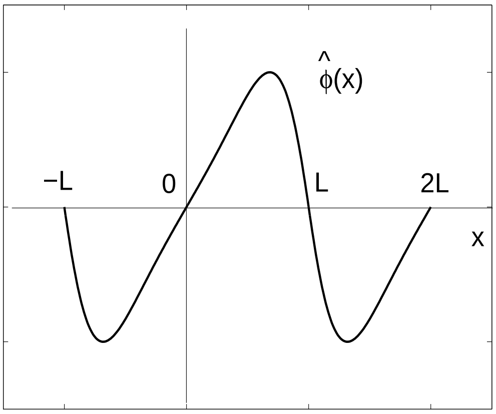
$$\tilde{u} = u - \left(g_0 - \frac{g_L - g_0}{L}\, x \right), \tag{16.12}$$

which satisfies both the Wave equation (16.3) and the zero boundary conditions

$$\tilde{u}(0,t)=0, \qquad \tilde{u}(L,t)=0.$$

Thus we set $g_0 = g_L = 0$ in (16.10) in what follows.

Now we will use a so-called *method of reflections*, where we claim that the solution of (16.3), (16.9), (16.10) (with $g_0 = g_L = 0$) is given by formula (16.6) with $\phi(x)$ and $\psi(x)$ being replaced by their anti-symmetric, $2L$-periodic extensions about points $x=0$ and $x=L$:



$$\hat{\phi}(x) = \begin{cases} \dots & \dots \\ -\phi(-x), & -L \le x \le 0 \\ \phi(x), & 0 \le x \le L \\ -\phi(2L-x), & L \le x \le 2L \\ \dots & \dots \end{cases} \tag{16.13}$$

and similarly for $\hat{\psi}(x)$.

Then the solution

$$u(x,t) = \frac{1}{2}\left(\hat{\phi}(x-ct) + \hat{\phi}(x+ct) \right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \hat{\psi}(s)\, ds \tag{16.14}$$

satisfies the PDE (16.3) (by virtue of (16.6)) and the initial condition (16.9) (by virtue of (16.13)). In a question for self-assessment you will be asked to verify that it also satisfies the zero boundary conditions at $x=0$ and $x=L$.

16.2 Wave equation as a system of first-order PDEs

Let us now present another point of view of the Wave equation. The numerical method developed in Lecture 17 will utilize this point of view.

If we denote

$$u_t = p, \qquad cu_x = q,$$

then Eq. (16.3) becomes

$$p_t - c q_x = 0\,. \tag{16.15a}$$

From the formula $u_{xt} = u_{tx}$ we obtain

$$q_t - c p_x = 0\,. \tag{16.15b}$$

In matrix form, these equations are written as

$$\frac{\partial}{\partial t}\begin{pmatrix} p \\ q \end{pmatrix} - c\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{16.16}$$

We proceed by diagonalizing the matrix in the above equation:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = S^{-1}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} S, \qquad S = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = S^{-1}. \tag{16.17}$$

Multiplying (16.16) on the left by S^{-1} and using (16.17), we arrive at a diagonal (i.e., decoupled) system of first-order heperbolic equations:

$$\frac{\partial}{\partial t}\begin{pmatrix} v \\ w \end{pmatrix} - \begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{16.18}$$

where

$$\begin{pmatrix} v \\ w \end{pmatrix} = S^{-1}\begin{pmatrix} p \\ q \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} p+q \\ p-q \end{pmatrix}. \tag{16.19}$$

In the component-by-component form, (16.18) is

$$v_t - c v_x = 0, \tag*{(16.18a)$'$}$$

$$w_t + c w_x = 0. \tag*{(16.18b)$'$}$$

In Appendix 2 we show that the general solutions of (16.18)$'$ are

$$v(x,t) = g(x+ct), \tag{16.20a}$$

$$w(x,t) = f(x-ct). \tag{16.20b}$$

Substituting these solutions into (16.19) and solving for p and q, we obtain

$$\begin{aligned} p = \quad u_t &= \tfrac{1}{\sqrt{2}}\left(g(x+ct) + f(x-ct)\right), \\ q = \quad c u_x &= \tfrac{1}{\sqrt{2}}\left(g(x+ct) - f(x-ct)\right). \end{aligned} \tag{16.21}$$

Integration of the latter equations yields

$$u(x,t) \;=\; F(x-ct) + G(x+ct), \tag*{(16.6)$'$}$$

where

$$F(\xi) = -\frac{1}{\sqrt{2}c}\int^{\xi} f(\tilde{\xi})\,d\tilde{\xi}, \qquad G(\eta) = \frac{1}{\sqrt{2}c}\int^{\eta} g(\tilde{\eta})\,d\tilde{\eta}\,. \tag{16.22}$$

Thus, we have reobtained the general solution (16.6) of the Wave equation.

As we have noted, the value of representing the Wave equation (16.3) in the form of a system of first-order equations, (16.16), is that in the next Lecture, we will develop methods of numerical solution of first-order hyperbolic PDEs. In preparation to this development, let us set up an initial-boundary value problem (IBVP) for the simplest first-order hyperbolic PDE,

$$w_t + cw_x = 0. \qquad (16.18\,\mathrm{b})'$$

In what follows, we assume $c > 0$ unless stated otherwise. As shown in Appendix 2, the solution of (16.18b)′ is given by (16.20b). Characteristics

$$x - ct = \xi = \text{const}, \quad \text{or} \quad t = \frac{1}{c}(x - \xi) \qquad (16.23)$$

of that equation are shown in the figure next to formulae (16.5). By looking at those characteristics, one sees that one can prescribe the IBVP for (16.18b)′ in two ways: either by an initial condition on the entire line,

$$w(x, t = 0) = \phi(x), \qquad -\infty < x < \infty \qquad (16.24)$$

or on the boundary of the first quadrant of the (x, t)-plane:

$$\begin{aligned} w(x, t = 0) &= \phi(x), \quad x \geq 0 \\ w(x = 0, t) &= g(t). \quad t \geq 0 \end{aligned} \qquad (16.25)$$

Then the initial and boundary (if applicable) values will propagate along the characteristics (16.23) and thereby determine the solution at any point inside the first quadrant ($x \geq 0,\ t \geq 0$). Note that the solution of (16.18b)′ *cannot be defined* in the second quadrant, ($x \leq 0,\ t \geq 0$), because the characteristics do not extend there.

16.3 Appendix 1: Derivation of D'Alambert's formula (16.8)

Substituting (16.6) into (16.7) and using the identities

$$F_t(x - ct) = -cF'(\xi) = -cF_x(\xi), \qquad G_t(x + ct) = cG'(\eta) = cG_x(\eta), \qquad (16.26)$$

where $F' \equiv dF/d\xi$ and $G' \equiv dG/d\eta$, one obtains:

$$F(x) + G(x) = \phi(x), \qquad -cF_x(x) + cG_x(x) = \psi(x)\,. \qquad (16.27)$$

Upon differentiating the first of these equations by x, one obtains a system of two equations for $F_x(x)$ and $G_x(x)$. In a homework problem you will be asked to verify that its solution integrated over x is

$$F(x) = \frac{1}{2}\left(\phi(x) - \frac{1}{c}\int_{-\infty}^{x} \psi(s)\,ds\right), \qquad G(x) = \frac{1}{2}\left(\phi(x) + \frac{1}{c}\int_{-\infty}^{x} \psi(s)\,ds\right). \qquad (16.28\text{a})$$

Hence

$$F(x{-}ct) = \frac{1}{2}\left(\phi(x - ct) - \frac{1}{c}\int_{-\infty}^{x-ct} \psi(s)\,ds\right), \qquad G(x{+}ct) = \frac{1}{2}\left(\phi(x + ct) + \frac{1}{c}\int_{-\infty}^{x+ct} \psi(s)\,ds\right). \qquad (16.28\text{b})$$

The substitution of (16.28b) into (16.6) yields (16.8).

16.4 Appendix 2: Solution of (16.18)′ is given by (16.20)

Let us begin by putting the PDE

$$w_t + c w_x = 0 \qquad (16.18\,\mathrm{b})'$$

in the form of an ODE. Consider a change of variables:

$$(x, t) \quad \rightarrow \quad (\xi = x - ct,\ \eta = x + ct). \qquad (16.29)$$

Using the Chain Rule for the function of several variables,

$$\frac{\partial}{\partial t} = \frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta}, \qquad \frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta}, \qquad (16.30)$$

we obtain:

$$\left(-c\frac{\partial}{\partial \xi} + c\frac{\partial}{\partial \eta}\right) w + c\left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right) w \;=\; 0 \qquad \Rightarrow \qquad w_\eta = 0. \qquad (16.31)$$

The last equation means that w depends only on ξ, which, in view of (16.23), implies (16.20b). Similarly, one shows that the solution of (16.18a)′ is given by (16.20a).

Thus, for any two "points" (x_1, t_1) and (x_2, t_2) in the (x, t)-plane that satisfy $x_1 - ct_1 = x_2 - ct_2$, the solution of (16.18b)′ satisfies

$$w(x_1, t_1) = w(x_2, t_2). \qquad (16.32)$$

16.5 Questions for self-assessment

1. Suppose one wants to develop a second-order accurate finite-difference scheme for a hyperbolic PDE. Which of the two ODE methods one should mimic the scheme after: the modified Euler or the Leap-frog?
2. What is the meaning of the solution (16.6)?
3. What is the meaning of each piece of the initial conditions (16.7)?
4. Where will the trick based on substitution (16.12) cause problems if the boundary conditions were to depend on time?
5. Verify the statements made after formula (16.14).
6. Why did we want to diagonalize the matrix in (16.16)?
7. Verify that you obtain (16.6)′ and (16.22) from (16.21).
8. Verify (16.31).