17 Method of characteristics for solving hyperbolic PDEs

In this lecture we will describe a method of numerical integration of hyperbolic PDEs which uses the fact that all solutions of such PDEs propagate along characteristics.

17.1 Method of characteristics for a single hyperbolic PDE

Let us start the discussion with the simplest, first-order hyperbolic PDE

$$w_t + c w_x = 0, \tag{17.1}$$

where we will take $c > 0$ for concreteness. For now, we assume that $c = \text{const}$; later on this restriction will be removed. The general solution of (17.1), derived in Appendix 2 of Lecture 16, is

$$w(x,t) = w(x - ct). \tag{17.2}$$

Thus, if the steps in x and t are related so that

$$\Delta x \;=\; c\,\Delta t, \tag{17.3}$$

then

$$w(x + \Delta x,\, t + \Delta t) \;=\; w\big(x + \Delta x - c(t + \Delta t)\big) \;=\; w(x - ct); \tag{17.4}$$

see also (16.32). This simply illustrates the fact that the solution does not change along the characteristic $x - ct = \xi$.

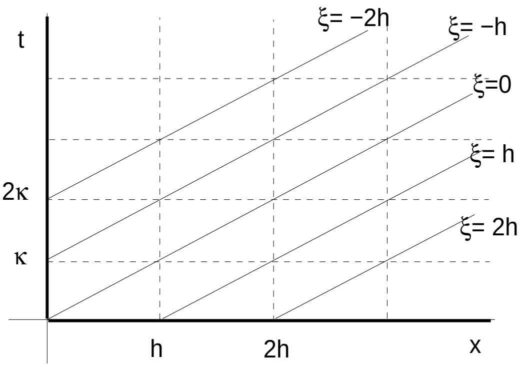
To put (17.4) at the foundation of a numerical method, consider the mesh

$$\begin{aligned} x_m &= m\,h, \quad m = 0, 1, 2, \ldots \\ t_n &= n\,\kappa, \quad\; n = 0, 1, 2, \ldots \end{aligned} \tag{17.5a}$$

where h and κ are related as per (17.3):

$$h \;=\; c\,\kappa. \tag{17.5b}$$

This is illustrated in the figure on the right.



The initial and boundary conditions for this problem are given by

$$\begin{aligned} w(x, t=0) &= \phi(x), \quad x \ge 0; \\ w(x=0, t) &= g(t), \quad\; t \ge 0. \end{aligned} \tag{16.25}$$

In the discretized form, they are:

$$\begin{aligned} W^0_m &= \phi(x_m), \quad m \ge 0; \\ W^n_0 &= g(t_n), \quad\;\; n \ge 0. \end{aligned} \tag{17.6}$$

(Obviously, we require $\phi(0) = g(0)$ for the boundary and initial conditions to be consistent with each other.) Then, according to (17.4), the solution at the node (m, n) with $m > 0$ and $n > 0$ is found as

$$W^n_m \;=\; \begin{cases} W^0_{m-n} = \phi(x_{m-n}), & m \ge n; \\ W^{n-m}_0 = g(t_{n-m}), & n \ge m. \end{cases} \tag{17.7}$$

This method, called the method of characteristics, can be generalized to the equation

$$w_t + c(x,t,w)\, w_x \;=\; f(x,t,w). \tag{17.8a}$$

For the sake of clarity, we will work out this generalization in two steps. First, we consider the case when $f(x,t,w) \equiv 0$, i.e.

$$w_t + c(x,t,w)\, w_x \;=\; 0. \tag{17.8b}$$

Then, making a change of variables

$$(x,\, t) \;\longrightarrow\; (\xi,\, t), \quad \text{where} \quad \xi = x - \int_0^t c\big(\, x(t'),\, t',\, w(x(t'),t')\,\big)\, dt', \tag{17.9}$$

and proceeding similarly[31] to Appendix 2 of Lecture 16, one can show that

$$w_t = 0 \qquad \Rightarrow \qquad w(x,t) = w(\xi) \;\text{ irrespective of a specific value of } t. \tag{17.10}$$

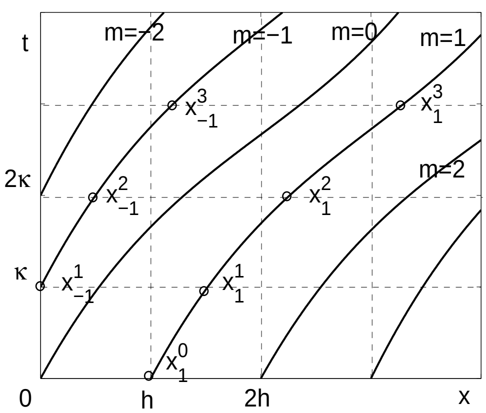
The equation for the characteristics $\xi = \text{const}$ of (17.8) is obtained by differentiating the expression

$$x - \int_0^t c\big(x(t'), t', w(x(t'),t')\big)\, dt' \;=\; \text{const}$$

(see (17.9)) with respect to t. The result is:

$$\frac{dx}{dt} = c(x,t,w), \qquad w = \text{const} \tag{17.11}$$

where the last condition ($w = \text{const}$) appears because along the characteristic, the solution w does not change (see (17.10)).



Note that unlike in the figure next to Eqs. (17.5), the characteristics corresponding to Eqs. (17.11) are *curved*, not straight, lines, as illustrated above.

The numerical solution of Eqs. (17.10) and (17.11) can be generated as follows. Let us denote x_m^n to be the grid point at the intersection of the time level $t = n\kappa$ and the characteristic $\xi = mh$ (see the figure above for an illustration). Note that this definition of x_m^n is **different from the definition of x_m in** (17.5a). Namely, there, x_m are fixed points of the spatial grid which are defined independently of the time grid. On the contrary, in scheme (17.12) below, x_m^n *moves along the m-th characteristic* and hence is different at each time level.

Continuing with setting up a scheme for (17.10) and (17.11), let W_m^n denote the value of w at the grid point x_m^n, i.e. $W_m^n \;=\; w\big(\xi = mh,\, t = n\kappa\big)$. Then:

$$\underline{n=0}: \qquad x_m^0 = mh, \qquad W_m^0 = \phi(mh), \qquad m \ge 0; \tag{17.12a}$$

$$\underline{n=1}: \qquad \begin{array}{ll} x_{-1}^1 = 0, & W_{-1}^1 = g(\kappa), \\ x_m^1 = x_m^0 + \displaystyle\int_0^\kappa c\big(x,t,W_m^0\big)\, dt, & W_m^1 = \phi(mh), \quad m \ge 0; \end{array} \tag{17.12b}$$

[31] E.g., $\partial_t = \partial_t \xi\, \partial_\xi + \partial_t t\, \partial_t = -c\, \partial_\xi + \partial_t$.

$$\underline{n \ge 2}: \qquad x^n_{-n} = 0, \qquad\qquad W^n_{-n} = g(n\kappa),$$

$$x^n_m = x^{n-1}_m + \int_{(n-1)\kappa}^{n\kappa} c\big(x, t, W^{n-1}_m\big)\, dt, \qquad \left\{ \begin{array}{ll} W^n_m = g(-m\kappa), & -(n-1) \le m < 0, \\ W^n_m = \phi(mh), & m \ge 0. \end{array} \right. \tag{17.12c}$$

Above, the expression $\int_{(n-1)\kappa}^{n\kappa} c(x,t,w)\, dt$ is a symbol denoting the result of integration of the ODE (17.11) from $t = (n-1)\kappa$ to $t = n\kappa$. This integration may be performed either analytically (if the problem so allows) or numerically using any of the numerical methods for ODEs. *Note that this integration is the* **only** *computation required in* (17.12); the rest of it is just the assignment of known values to the grid points at each time level.

Let us emphasize the meaning of scheme (17.12). First, it computes the values x^n_m along the respective characteristics for each m as per the first equation in (17.11). Then, the value of w is kept constant along each characteristic, as specified by (17.10).

Remark If one wants to keep the number of grid points at each time level of (17.12) the same (say, $(M+1)$), then one needs to "chop off" the right-most point at every time step. Recall also that one cannot prescribe a boundary condition at the right boundary $x = Mh$.

Let us illustrate the solution of (17.8b) by Eqs. (17.12) for the so-called *shock wave equation*[32]

$$u_t + u\, u_x \;=\; 0, \tag{17.13}$$

which arises in a great many applications (e.g., in gas dynamics or in traffic flow modelling). As an initial condition, let us take

$$u(x,0) \;\equiv\; \phi(x) \;=\; \left\{ \begin{array}{ll} a\, \sin^2 \pi x, & 0 \le x \le 1, \\ 0, & \text{otherwise,} \end{array} \right. \tag{17.14}$$

where a is some constant. We will consider this problem on the infinite line, $x \in (-\infty, \infty)$, but in our numerical solution will only follow points where $u \neq 0$.

According to Eqs. (17.10) and (17.9), the solution of problem (17.13), (17.14) is given by an implicit formula

$$u = \phi\left(x - \int_0^t u(x(t'), t')\, dt' \right), \tag{17.15}$$

where we have used the initial condition $u(x, t=0) = \phi(x)$. Recall that in (17.15), $x(t)$ stands for the equation of one given characteristic; in other words, the integral is computed along that characteristic. We will now show that characteristics of (17.13) have a special form that allow the integral in (17.15) to be be simplified. Indeed, the equation for the characteristics of (17.13) follows from (17.11):

$$\frac{dx}{dt} \;=\; u, \qquad u \;=\; \text{const.} \tag{17.16}$$

Thus, since the u *on the right-hand side* of (17.15) is constant along the characteristic, then (17.15) reduces to

$$u = \phi(x - ut). \tag{17.15$'$}$$

This is now an implicit *algebraic* equation for u which, in principle, can be solved for each pair (x, t).

[32]Another name of this equation, or, more precisely, of the more general equation $u_t + c(u)\, u_x = 0$, is the "simple wave" equation, with the adjective "simple" originating from physical considerations.

To obtain a numerical solution of (17.13), (17.14) *explicitly*, we use a modification of (17.12) which would allow us to keep track of only those grid points where $u \neq 0$. The correponding scheme on such a *moving grid* is:

$$\underline{n=0}: \qquad x_m^0 = mh, \qquad U_m^0 = \phi(mh); \tag{17.17a}$$

$$\underline{n=1}: \qquad x_m^1 = x_m^0 + U_m^0\kappa, \qquad U_m^1 = \phi(mh); \tag{17.17b}$$

(where the meaning of index m is clarified in (17.17d) below)

$$\underline{n \geq 2}: \qquad x_m^n = x_m^{n-1} + U_m^{n-1}\kappa, \qquad U_m^n = U_m^{n-1} = \ldots = U_m^0. \tag{17.17c}$$

Note that in all these equations,

$$m = 0,\, 1,\, \ldots\, ,\, M, \quad \text{and} \quad h = 1/M \;\; \text{(see (17.14))}, \tag{17.17d}$$

so that a particular value of m labels the characteristic emanating at point $(x = mh,\, t = 0)$. This way of labeling is illustrated in the figure next to Eq. (17.11). It defines a grid which moves to the right (given that the initial velocity $\phi(x) \geq 0$). Also, at each time level except the one at $t = 0$, the internode spacing along x is not uniform. Note that (17.17) is the discretized form of the *exact* analytical solution (17.15′). You will be asked to plot solution (17.17) in a homework problem.

Let us now return to Eq. (17.8) with $f \not\equiv 0$. Similarly to (17.10), one then obtains

$$w_t \;=\; f(\xi, t, w), \tag{17.18a}$$

where $f(\xi, t, w)$ is obtained from $f(x, t, w)$ by the change of variables (17.9). (For example, if $f(x, t) = x + t^2$ and $c = 3$, then $f(\xi, t) = \xi + 3t + t^2$.) Equation (17.18a) says that $w(\xi,\, t)$ is no longer a constant along a characteristic

$$\xi = \text{const} \tag{17.18b}$$

but instead varies along it in the prescribed manner. When solving (17.18a), ξ should be considered as a constant parameter. To find the equation of the characteristics, one needs to solve the first equation in (17.11) where instead of the second equation in (17.11), i.e. $w = \text{const}$, one now needs to use (17.18a). Thus, the solution of the original PDE (17.8a) reduces to the solution of two coupled ODEs: (17.18) and

$$\frac{dx}{dt} \;=\; c(\xi,\, t,\, w), \qquad x(t=0) = \xi, \tag{17.19}$$

where $c(\xi, t, w)$ is obtained from $c(x, t, w)$ by the change of variables (17.9) (see the clarification after (17.18a)). An implementation of the solution of (17.18) and (17.19) that assumes the boundary conditions (17.6) is given below:

$$\underline{n=0}: \qquad x_m^0 = mh, \quad \xi_m = x_m^0, \qquad W_m^0 = \phi(mh); \tag{17.20a}$$

$$\underline{n=1}: \qquad x_{-1}^1 = 0, \qquad W_{-1}^1 = g(\kappa),$$

$$\begin{pmatrix} x_m^1 \\ W_m^1 \end{pmatrix} = \begin{pmatrix} x_m^0 \\ W_m^0 \end{pmatrix} + \int_0^\kappa \begin{pmatrix} c\big(\xi_m,\, t,\, w_m(t)\big) \\ f\big(\xi_m,\, t,\, w_m(t)\big) \end{pmatrix} dt, \quad m \geq 0; \tag{17.20b}$$

$$\underline{n \geq 2}: \qquad x^n_{-n} = 0, \qquad W^n_{-n} = g(n\kappa),$$

$$\begin{pmatrix} x^n_m \\ W^n_m \end{pmatrix} = \begin{pmatrix} x^{n-1}_m \\ W^{n-1}_m \end{pmatrix} + \int_{(n-1)\kappa}^{n\kappa} \begin{pmatrix} c\big(\xi_m,\, t,\, w_m(t)\big) \\ f\big(\xi_m,\, t,\, w_m(t)\big) \end{pmatrix} dt, \quad m \geq -n+1. \tag{17.20c}$$

Here the expression

$$\int_{(n-1)\kappa}^{n\kappa} \begin{pmatrix} c\big(\xi_m,\, t,\, w_m(t)\big) \\ f\big(\xi_m,\, t,\, w_m(t)\big) \end{pmatrix} dt$$

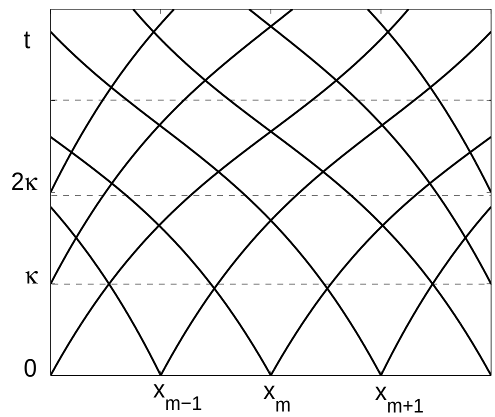
is a symbol that denotes the result of integration of the coupled ODEs (17.18) and (17.19). In practice, this integration can be done numerically by any suitable ODE method. Also, $w_m(t)$ above means the solution along the characteristic $\xi = \xi_m$ (see (17.20a)).

The meaning of scheme (17.20) is the following. As previously scheme (17.12), it computes the curves of characteristics $x_m(t) = \xi_m$ as per (17.19). However, unlike (17.12), now the value of w is not constant along each characteristic but instead varies according to (17.18). Note that now the equations for the characteristic and for the solution w are coupled and need to be solved simultaneously.

17.2 Method of characteristics for a system of hyperbolic PDEs

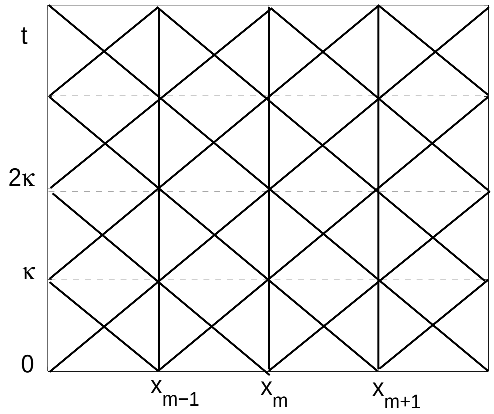
In this section, we will first point out technical diffuculties that can arise when using the method of charateristics for a system of PDEs. Then we will work out an example where those difficulties do *not* occur.

If we attempt to generalize the approach that led to schemes (17.12) and (17.20) to the case where one has a coupled system of *two* PDEs with intersecting families of curved (i.e., not straight-line) characteristics, one is likely to encounter a problem depicted in the figure on the right. Namely, suppose the characteristics of the two families are chosen to intersect at level $t = 0$. In the figure, the intersection points are x_{m-1}, x_m, x_{m+1}, etc. However, these characteristics no longer intersect at subsequent time levels; this is especially visible at levels $t = 2\kappa$ and $t = 3\kappa$.



An analogous problem can also occur if one has *three or more* characteristics, even if they are straight lines. The only case where this will *not* occur is where all the characteristics can be chosen to intersect at uniformly spaced points at each level. An example of such a special situation is shown on the right. Note that the vertical characteristics are just the lines

$$\frac{dx}{dt} = 0 \quad \Rightarrow \quad x(t) = \xi_m \equiv \text{const.} \qquad (17.21)$$



Characteristrics (17.21) occur whenever the system of PDEs includes an equation

$$w_{j,t} = f\big(x, t, \vec{\mathbf{w}}\big). \qquad (17.22)$$

You will be asked to verify this in a QSA.

A way around this issue is to interpolate the values of the solution at each time level. For example, suppose one is to solve a system of two PDEs for w_1 and w_2 on the segment $0 \le x \le 1$, with the characteristics for w_1 (w_2) going northeast (northwest). Let there be $(M-1)$ internal points, $x_m = mh$, $m = 1, \ldots, (M-1)$ at the initial time level $t_0 = 0$. Suppose that the characteristics for w_j, $j = 1, 2$, intersect the next time level $t_1 = \kappa$ at points $x_m^{(j)}$. Then one can interpolate the set of values $w_m^{(j)}$ from the respective nonuniform grid $x_m^{(j)}$ onto the same grid x_m as at the initial time level. This interpolation process is then repeated at every time level.

Matlab's command to interpolate a vector `y` from a grid defined by a vector `x` (such that `length(x)=length(y)`) onto a vector `xx` is:
`yy = spline( x, y, xx)` .

We will now work out a solution of a system of two PDEs with straight-line charactristics:

$$\begin{aligned} w_{1,t} + c\, w_{1,x} &= f_1(x, t, w_1, w_2), \\ w_{2,t} - c\, w_{2,x} &= f_2(x, t, w_1, w_2), \\ w_j(x, 0) = \phi_j(x), &\qquad 0 \le x \le 1, \quad j = 1, 2 \\ w_1(0, t) = g_1(t), &\quad w_2(1, t) = g_2(t), \qquad t \ge 0. \end{aligned} \qquad (17.23)$$

The two characteristic directions of (17.23) are

$$\xi_j = x - c_j t, \qquad j = 1, 2; \qquad c_1 = c, \quad c_2 = -c. \qquad (17.24)$$

If f_j for $j = 1$ and/or 2 in (17.23) vanishes, then the respective w_j will not change along its characteristic ξ_j. Therefore, for $f_j \ne 0$, it is convenient to calculate the change of w_j *along the characteristic* ξ_j. Then, similarly to (17.18a), we can write the first two equations of (17.23) as

$$w_{j,t} = f_j\big(\,\xi_j + c_j t,\ t,\ w_1(\xi_1, t),\ w_2(\xi_2, t)\,\big) \quad \text{along } \xi_j = \text{const}, \qquad j = 1, 2. \qquad (17.25)$$

Note that while integrating, say, the equation with $j = 1$, the argument ξ_2 of w_2 should *not* be considered as constant. At the moment, this prescription is rather vague, but later on we will present a specific example of how it can be implemented.

The formal numerical implementation of the solution of (17.25) is given below on the grid (17.5) where the maximum value of $m = M$ corresponds to the right boundary $x = 1$. Note that this grid is *stationary* and hence is *different from the moving grids* used in schemes (17.12), (17.17), and (17.20). In particular, in this stationary grid, m does *not* label a particular characteristic.

The scheme for (17.25) is:

$$\underline{n=0}: \qquad (\xi_j)^0_m = x_m, \quad (W_j)^0_m = \phi_j(mh), \quad j = 1, 2; \tag{17.26a}$$

$$\begin{aligned}
\underline{n \ge 1}: \qquad & (\xi_1)^n_m = x_m - c_1\kappa n \;\equiv\; (m-n)h, \qquad \text{(see (17.5a,b))} \\
& (W_1)^n_0 = g_1(n\kappa), \\
& (W_1)^n_m = (W_1)^{n-1}_{m-1} + \int_{(n-1)\kappa}^{n\kappa} f_1\big((\xi_1)^{n-1}_{m-1} + c_1 t,\, t,\, W_1,\, W_2\big)\, dt, \qquad m = 1, \ldots, M; \\
& (\xi_2)^n_m = x_m - c_2\kappa n \;\equiv\; (m+n)h, \\
& (W_2)^n_M = g_2(n\kappa), \\
& (W_2)^n_m = (W_2)^{n-1}_{m+1} + \int_{(n-1)\kappa}^{n\kappa} f_2\big((\xi_2)^{n-1}_{m+1} + c_2 t,\, t,\, W_1,\, W_2\big)\, dt, \qquad m = 0, \ldots, M-1.
\end{aligned} \tag{17.26b}$$

Note that with the step sizes along the temporal and spatial coordinates being related by (17.5b), the values of ξ_1 and ξ_2 stay constant along the lines $m - n = \text{const}$ and $m + n = \text{const}$, respectively.

To turn scheme (17.26) into a useful tool, we need to specify how the integrals

$$\int_{(n-1)\kappa}^{n\kappa} f_j\big((\xi_j)^{n-1}_{m+(-1)^j} + c_j t,\, t,\, W_1,\, W_2\big)\, dt, \qquad j = 1, 2$$

can be computed. Recall that these integrals are just the symbols denoting the increment of the solutions of (17.25) from $t = (n-1)\kappa$ to $t = n\kappa$ along the respective characteristic $\xi_j = \text{const}$. Below we show how this can be done by the modified explicit Euler method. We will write the equations first and then will comment on their meaning.

$$\begin{aligned}
\bar W_1 &= (W_1)^{n-1}_{m-1} \;+\; \kappa\, f_1\Big(\, (\xi_1)^{n-1}_{m-1} + c_1\kappa(n-1),\; c\kappa(n-1),\; (W_1)^{n-1}_{m-1},\; (W_2)^{n-1}_{m-1}\Big), \\
\bar W_2 &= (W_2)^{n-1}_{m+1} \;+\; \kappa\, f_2\Big(\, (\xi_2)^{n-1}_{m+1} + c_2\kappa(n-1),\; c\kappa(n-1),\; (W_1)^{n-1}_{m+1},\; (W_2)^{n-1}_{m+1}\Big);
\end{aligned} \tag{17.27a}$$

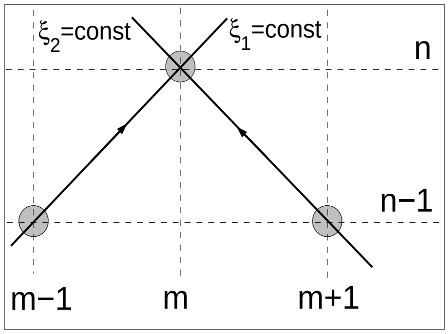
$$\begin{aligned}
(W_1)^n_m &= \frac{1}{2}\Big[(W_1)^{n-1}_{m-1} + \bar W_1 \;+\; \kappa\, f_1\Big(\, (\xi_1)^n_m + c_1\kappa n,\; c\kappa n,\; \bar W_1,\; \bar W_2\Big)\Big], \\
(W_2)^n_m &= \frac{1}{2}\Big[(W_2)^{n-1}_{m+1} + \bar W_2 \;+\; \kappa\, f_2\Big(\, (\xi_2)^n_m + c_2\kappa n,\; c\kappa n,\; \bar W_1,\; \bar W_2\Big)\Big].
\end{aligned} \tag{17.27b}$$

Note that the notations $(\xi_1)^{n-1}_{m-1} + c_1\kappa(n-1)$ and $(\xi_2)^{n-1}_{m+1} + c_2\kappa(n-1)$ in (17.27a) have been used only to mimic the corresponding terms in (17.25). Those terms, as evident from the first two equations of (17.23) and from (17.25), must equal x_{m-1} and x_{m+1} for $j = 1$ and $j = 2$, respectively. Indeed:

$$
\begin{aligned}
(\xi_1)^{n-1}_{m-1} + c_1\kappa(n-1) &= \big(h(m-1) - c\kappa(n-1)\big) + c\kappa(n-1) = x_{m-1},\\
(\xi_2)^{n-1}_{m+1} + c_2\kappa(n-1) &= \big(h(m+1) + c\kappa(n-1)\big) - c\kappa(n-1) = x_{m+1},
\end{aligned}
$$

where we have used the equations for $(\xi_j)^n_m$ from (17.26b). Similarly, $(\xi_j)^n_m + c_j\kappa n$ in (17.27b) equal x_m for both $j = 1$ and 2.

The meaning of the first equation in (17.27a) is the following. The change of W_1 is computed *along the characteristic* $\xi_1 = (\xi_1)^{n-1}_{m-1}$ by the simple Euler approximation, whereby *all arguments of* f_1 are evaluated at the "starting" node ($x = x_{m-1}$, $t = t_{n-1}$). Since, as we have said, this change occurs along the characteristic $\xi_1 = (\xi_1)^{n-1}_{m-1}$, which is labeled "$\xi_1 = \text{const}$" in the figure on the right, then the "final" node of this step is ($x = x_m$, $t = t_n$).



Similarly, the change of W_2 in (17.27a) is computed along the characteristic $\xi_2 = (\xi_2)^{n-1}_{m+1}$ by the simple Euler approximation; hence all arguments of f_2 are evaluated at the "starting" node ($x = x_{m+1}$, $t = t_{n-1}$) for *that* characteristic (which is labeled "$\xi_2 = \text{const}$" in the figure above.) The step along this characteristic ends at the same node ($x = x_m$, $t = t_n$).

Finally, the equations in (17.27b) are the standard "corrector" equations of the explicit modified Euler method.

Scheme (17.26), (17.27) can be straightforwardly generalized for more than two coupled first-order hyperbolic PDEs, as long as all the characteristics can be chosen to intersect at uniformly spaced points at each time level. An example of that situation is shown in the figure next to Eq. (17.21). Extending the scheme to use a Runge–Kutta type method of order higher than two (which is the order of the modified explicit Euler method), or to use any other method (say, leap-frog), also appears to be straightforward.

17.3 Questions for self-assessment

1. What is the meaning of scheme (17.7)?
2. Where can one specify a boundary condition for Eqs. (17.8) and where can one not?
3. Why is $w = \text{const}$ in (17.11)?
4. What is the meaning of scheme (17.12)?
5. What does the expression $\displaystyle\int_{(n-1)\kappa}^{n\kappa} c(x(t'), t', w)\, dt'$ in (17.12) stand for?
6. Explain where solution (17.15) comes from and then how it is reduced to (17.15′).
7. What is the meaning of scheme (17.20)?

8. Describe a technical problem that is likely to occur when solving a system of coupled PDEs by the method of chatacteristics. How can this problem be overcome?

9. Verify the statement found after Eq. (17.22).

10. What is the difference between the grids used in schemes (17.12) and (17.20), on one hand, and in scheme (17.26), on the other?

11. Why is W_2 in the first line of (17.27a) evaluated at x_{m-1}? Why is W_1 in the second line of (17.27a) evaluated at x_{m+1}?