6 Boundary-value problems (BVPs): Introduction

A typical BVP consists of an ODE and a set of conditions that its solution has to satisfy *at both ends* of a certain interval $[a,\, b]$. For example:

$$y'' = f(x, y, y'), \qquad y(a) = \alpha, \quad y(b) = \beta\,. \tag{6.1}$$

Now, let us recall that for IVPs

$$y' = f(x, y), \qquad y(x_0) = y_0,$$

we saw (in Lecture 0) that there are certain conditions on the function $f(x, y)$ which would guarantee that the solution $y(x)$ of the IVP exists and is unique.

The situation with BVPs is considerably more complicated. Namely, relatively few theorems exist that can guarantee existence and uniqueness of a solution to a BVP. Below we will state, without proof, two of such theorems. In this course we will always assume that $f(x, y, y')$ in (6.1) and in similar BVPs is a continuous function of its arguments.

Theorem 6.1 Consider a BVP of a special form:

$$y'' = f(x, y), \qquad y(a) = \alpha, \quad y(b) = \beta\,. \tag{6.2}$$

(Note that unlike (6.1), the function f in (6.2) is assumed not to depend on y'.)
If $\partial f/\partial y > 0$ for all $x \in [a,\, b]$ and all values of the solution y, then the solution $y(x)$ to the BVP (6.2) exists and is unique.

This Theorem is not very useful for a general nonlinear function $f(x, y)$, because we do not know the solution $y(x)$ and hence cannot always determine whether $\partial f/\partial y > 0$ or < 0 for the specific solution that we are going to obtain. Sometimes, however, as, e.g., when $f(x, y) = f_0(x) + y^3$, we *are* guaranteed that $\partial f/\partial y > 0$ for any y and hence the solution of the corresponding BVP does exist and is unique. Another useful and common case is when the BVP is *linear*. In this case, we have the following result.

Theorem 6.2 Consider a linear BVP:

$$y'' + P(x)y' + Q(x)y = R(x), \qquad y(a) = \alpha, \quad y(b) = \beta\,. \tag{6.3}$$

Let the coefficients $P(x)$, $Q(x)$, and $R(x)$ be continuous on $[a,\, b]$ and, *in addition*, let $Q(x) \le 0$ on $[a,\, b]$. Then the BVP (6.3) has a unique solution.

Note that in addition to the *Dirichlet* boundary conditions considered above, i.e.

$$y(a) = \alpha, \quad y(b) = \beta, \qquad \text{(Dirichlet b.c.)}$$

there may be also boundary conditions for the derivative, called the Neumann boundary conditions:

$$y'(a) = \alpha, \quad y'(b) = \beta. \qquad \text{(Neumann b.c.)}$$

Also, the boundary consitions may be of the mixed type:

$$A_1 y(a) + A_2 y'(a) = \alpha, \quad B_1 y(b) + B_2 y'(b) = \beta. \qquad \text{(mixed b.c.)}$$

Boundary conditions that involve values of the solution at both boundaries are also possible; e.g., periodic boundary conditions:

$$y(a) = y(b), \quad y'(a) = y'(b); \qquad \text{(periodic b.c.)}$$

however, we will only consider the boundary conditions of the first three types (Dirichlet, Neumann, and mixed) in this course.

Note that Theorems 6.1 and 6.2 are stated specifically for the Dirichlet boundary conditions. For the Neumann boundary conditions, they are not valid. Specifically, in the case of Theorem 6.2 applied to the linear ODE (6.3) with Neumann boundary conditions, the solution will exist but may only be unique up to an arbitrary constant. (That is, if $y(x)$ is a solution, then so is $y(x) + C$, where C is any constant.)

A large collection of other theorems about existence and uniqueness of solutions of linear and nonlinear BVPs can be found in a very readable book by P.B. Bailey, L.F. Shampine, and P.E. Waltman, "Nonlinear two-point boundary value problems," Ser.: Mathematics in science and engineering, vol. 44 (Academic Press 1968).

Unless the BVP satisfies the conditions of Theorems 6.1 or 6.2, it is not guaranteed to have a unique solution. In fact, depending on *the specific combination of the ODE and the boundary conditions*, the BVP may have: (i) no solutions, (ii) one solution, (iii) a finite number of solutions, or (iv) infinitely many solutions. Possibility (iii) can take place only for nonlinear BVPs, while the other three possibilities can take place for both linear and nonlinear BVPs. In the remander of this lecture, we will focus on linear BVPs.

Thus, a linear BVP can have 0 solutions, 1 solution, or the ∞ of solutions. This is similar to how a matrix equation

$$M\vec{\mathbf{x}} = \vec{\mathbf{b}} \tag{6.4}$$

can have 0 solutions, 1 solution, or the ∞ of solutions, depending on whether the matrix M is singular or not. The reason behind this similarity will become apparent as we proceed, an early indication of this reason appearing later in this lecture and more evidence appearing in the subsequent lectures. Below we give three examples where the BVP does *not* satisfy the conditions of Theorem 6.2, and each of the above three possibilities is realized.

$$y'' + \pi^2 y = 1, \qquad y(0) = 0, \quad y(1) = 0 \qquad \text{(Problem I)}$$

has 0 solutions.

$$y'' + \pi^2 y = 1, \qquad y(0) = 0, \quad y'(1) = 1 \qquad \text{(Problem II)}$$

has exactly 1 solution.

$$y'' + \pi^2 y = 1, \qquad y(0) = 0, \quad y(1) = \frac{2}{\pi^2} \qquad \text{(Problem III)}$$

has the ∞ of solutions.

Below we demonstrate that the above statements about the numbers of solutions in Problems I–III are indeed correct.

One can verify that the general solution of the ODE

$$y'' + \pi^2 y = 1 \tag{6.5}$$

is

$$y \;=\; A\sin\pi x \;+\; B\cos\pi x \;+\; \frac{1}{\pi^2}\,. \tag{6.6}$$

The constants A and B are determined by the boundary conditions. Namely, substituting the solution (6.6) into the boundary conditions of Problem I above, we have:

$$y(0) = 0 \quad \Rightarrow \quad B + \frac{1}{\pi^2} = 0; \qquad y(1) = 0 \quad \Rightarrow \quad -B + \frac{1}{\pi^2} = 0\,; \tag{6.7}$$

hence no such B (and hence no pair A, B) exists. Note that the above equations can be written as a linear system for the coefficients A and B:

$$\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} -\frac{1}{\pi^2} \\ -\frac{1}{\pi^2} \end{pmatrix}. \tag{6.8}$$

The coefficient matrix in (6.8) is singular and, in addition, the vector on the r.h.s. does not belong to the range (column space) of this coefficient matrix. Therefore, the linear system has no solution, as we have stated above.

Substituting now the solution (6.6) of the ODE (6.5) into the boundary conditions of Problem II, we arrive at the following linear system for A and B:

$$y(0) = 0 \quad \Rightarrow \quad B + \frac{1}{\pi^2} = 0; \qquad y'(1) = 1 \quad \Rightarrow \quad -\pi A = 1\,, \tag{6.9}$$

or in the matrix form,

$$\begin{pmatrix} 0 & 1 \\ -\pi & 0 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} -\frac{1}{\pi^2} \\ 1 \end{pmatrix}. \tag{6.10}$$

This system obviously has the unique solution $A = -\frac{1}{\pi}$, $B = -\frac{1}{\pi^2}$; note that the matrix in the linear system (6.10) is nonsingular.

Finally, substituting (6.6) into the boundary conditions of Problem III, one finds:

$$y(0) = 0 \quad \Rightarrow \quad B + \frac{1}{\pi^2} = 0; \qquad y(1) = \frac{2}{\pi^2} \quad \Rightarrow \quad -B + \frac{1}{\pi^2} = \frac{2}{\pi^2}\,, \tag{6.11}$$

or in the matrix form,

$$\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} -\frac{1}{\pi^2} \\ \frac{1}{\pi^2} \end{pmatrix}. \tag{6.12}$$

The solution of (6.12) is: $A =$ arbitrary, $B = -\frac{1}{\pi^2}$. Although the matrix in (6.12) is singular, the vector on the r.h.s. of this equation belongs to the column space of this matrix (that is, it can be written as a linear combination of the columns), and hence the linear system in (6.12) has infinitely many solutions.

The above simple examples illustrate a connection between linear BVPs and systems of linear equations. We can use this connection to formulate an analogue of the well-known theorem for linear systems, namely:

Theorem in Linear Algebra: The linear system (6.4) has a unique solution if and only if the matrix M is nonsingular.
Equivalently, either the linear system has a unique solution, or the *homogeneous* linear system

$$M\vec{\mathbf{x}} = \vec{\mathbf{0}}$$

has nontrivial solutions. (Recall that the second part of the previous sentence is one of the definitions of a singular matrix.)

Similarly, for BVPs we have

The Alternative Principle for BVPs: Either the homogeneous linear BVP (i.e. the one with both the r.h.s. $R(x) = 0$ and zero boundary conditions) has nontrivial solutions, or the original BVP has a unique solution.

In a homework problem, you will be asked to verify this principle for Problems I–III.

To conclude this introduction to BVPs, let us again consider Problem I and exhibit a "danger" associated with that case. By itself, the fact that the BVP in Problem I has no solutions, is neither good nor bad; it is simply the fact of life. However, suppose that the coefficients in this problem are known only approximately (for example, because of the round-off error). Then the matrix in (6.8) is no longer singular (in general), but *almost* singular. This, in turn, makes the linear system (6.8) *ill-conditioned*: a tiny change of the vector on the r.h.s. will generically lead to a large change in the solution. Thus, any numerical results obtained for such a system will not be reliable. In a homework problem, you will be asked to consider a specific example illustrating this case.

Questions for self-assessment

1. Can you say anything about the existence and uniqueness of solution of the BVP

$$y'' = \arctan y, \qquad y(-1) = -\pi, \quad y'(1) = \pi \text{ ?}$$

2. Same question about the BVP

$$y'' + (\arctan x)\, y' + (\sin x)\, y = 1, \qquad y(-\pi) = -1, \quad y(0) = 1 \text{ .}$$

3. Same question about the BVP

$$y'' + (\arctan x)\, y' - (\sin x)\, y = 1, \qquad y'(0) = 0, \quad y'(1) = 1 \text{ .}$$

4. How many solutions are possible for a BVP? What if the BVP is linear?
5. Verify that (6.6) is the solution of (6.5).
6. Verify each of the equations (6.7)–(6.12).
7. Verify that the vector on the r.h.s. of (6.12) belongs to the range (column space) of the matrix on the l.h.s..
8. State the Alternative Principle for the BVPs.
9. What is the danger associated with the case when the BVP has no solution?